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This report gives information regarding the mathematical classroom activities for the first six grades at Estabrook School from March 1964 to June 1965. A brief progress report is given regarding the instruction provided to teach such concepts as addition and subtraction, symmetry transformations of squares, open sentences and graphing, inequalities, and vector addition in two dimensions. [Not available in hard copy due to marginal legibility of original document]. (RP)

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CCSM
Feasibility Study No. 33
Lomon

Cambridge Conference on School Mathematics
Estabrook School, Lexington

First-Fifth Grades

Progress Report for March, 1964

March

All three classes started in February were continued on a two-session-per-week basis. The third-fourth grade class (△ team) was doubled to comprise about thirty students. There has been progress in learning the subject matter and in formulating it. Motivation was found to be lacking in the symmetry transformation unit (△ team) and modification to further motivation is being considered. The use of tables and graphs in the probability unit (K team) seems to have had a positive influence on the handling of science projects by the same students.

At the invitation of Professor R. Davis of Webster College, I spent two days in St. Louis observing classes of the Madison Project. Of particular interest was the good performance by a lowest quartile fourth grade class with algebraic identities. I have obtained Madison Project films for viewing by interested Estabrook teachers. Madison Project paid my travelling expenses.

April

The first grade class has developed a "stick line" concept (a straight line with a "starting point - S" marked along it). They compare, add and subtract sticks on the "stick line". They have subtracted larger sticks from smaller sticks, ending up on the other side of the starting point. They have a beginning (b) and end (e) marked on each stick and an arrow from b to e. For any operation they place b either at the starting point or at the end e of the previous stick. For addition the arrow points to the door (right) and for subtraction to the window (left). The concept of a negative stick (arrow from e to b) is to be developed. We are beginning comparisons that lead to commutativity and associativity.

The third grade class progressed to symmetry transformations of squares. They discerned that there were four symmetry rotations to return to the first square, and four possible "twist" axes, compared to three for equilateral triangles. They observed the non-commutativity, and closure of the operations, and the existence of inverses and of an identity. They developed rea-

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sonable skill in the operations and in observing the relations that appear.

The fifth grade discussed the behavior of the most probable number of ups in the thumb tack experiment, and the range as a function T = U + D, the sample set size. They also examined the behavior of these same quantitities divided by T. The $1/\sqrt{n}$ progress of $\sqrt{}/T$ has been noted. They have started a random number experiment and are discussing what they expect to observe.

May

The grade 1 class developed facility in adding and subtracting on the "stick line" by considering the daily use of a week's supply of chalk, including "borrowing" chalk. Generalizing to the number line the "mailman" game was played (delivery of checks and bills). A competition (for the temporary custody of stuffed animals) was held requiring the correct and fast realization of having collected more than a certain positive amount of money (to buy the animals) or having incurred a certain amount of debt (must sell the animals). In dealing with dollars we have kept to integers so far. The facility of the class is now very good and we will move on to fractional points. Not all children like to think they are playing games!

The grade 3 class has gone all the way through translational symmetries of one and two dimensional infinite arrays of dots (a single spacing parameter). There was a "moment of truth" in the realization that an infinite array allowed symmetry motions that infinite arrays do not allow, no matter how large they are. They were intrigued by the application to the infinite hotel with one more guest problem. The development of the co-ordinate system was a by-product.

The grade 5 class progressed slowly. They had to learn "open sentences" and graphing before we could find mathematical models that would fit their "peak" and "range" data and allow predictions. They have now accomplished this, and a little appreciation of "random walk" and "random number" as well.

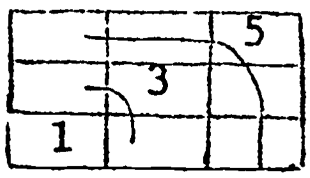
June

Classes at Estabrook continued until June 18, and the last classes in our units were held June 17. In grades 3 and 5, having come to a natural stopping point in the main units before the end of the term, new material was introduced in the last two weeks.

In grade 1 we played the "paper-stone-scissors game and examined it as an example in which our 3 stick rule did not work. There were frequent requests to repeat the game. We developed the $\frac{1}{2}$ points on the stick line, first by sandwiching a point (it is < 3 and > 2) then by defining $\frac{1}{2} + \frac{1}{2} = 1$ and cutting a paper stick until it satisfied that criterion. All the $\frac{1}{2}$ points were then constructed by addition and subtraction. The 1/3 points were defined analogously.

In grade 3 the commutativity and closure of two dimensional translations was established, and the need for two unit vectors and their inverses as a basis was "uncovered". The three dimensional analog was discussed. In one and two dimensions the "push" symmetries for arrays of two types of lattice points or two lattice spacings were discussed. The symmetry of people led to the discussion of mirror symmetries. We then changed topics and considered the sum of odd numbers. We first obtained $1 + 3 + \cdots + (2 \times \square - 1) = \square \times \square$

geometrically



Then discussed the proof by mathematical induction. The former brought up suggestions of putting cubes in three dimensions. The latter brought out some discussion of the logic and a very intense use of the C-A-D laws.

In fifth grade we graphed functions of linear and quadratic form and found $\triangle = 3/4 \times \square$ was a good representation of P (most probable value) vs. T (sample size), while $\triangle \times \triangle = 2 \times \square$ represented our Υ (full range) vs. T data reasonably. We used these equations for extrapolation and interpolation. We found the range formula to be applicable to the fluctuations of the California Republican primary vote as based on the early "profile" forecast. The "experiment independence" of the range formula was brought out by this and another thumbtack experiment. One student made a random walk plot of thumbtack data. The envelope was well approximated by our quadratic formula. We then turned to a discussion of a thrown object at the earth's surface. Newton's first and second laws were discussed. Using the independence of the vertical and horizontal motion we developed the following sequence of results. If a ball is thrown horizontally with an initial velocity of 2 ft/sec then

▱ = 2 × ○, where ▱ is horizontal distance(feet)

and ○ is time (sec)

◇ = 32 × ○ where ◇ is vertical velocity after time ○ (ft/sec)

▭ = ½ × ◇ = 16 × ○, where ▭ is the average vertical velocity from zero time to ○.

Then as an estimate (exact as it turns out)

△ = ▭ × ○ = 16 × ○ × ○ where △ is the vertical distance

Then by substitution △ = 4 × □ × □

We plotted the last and asked what a "harder throw" would do, and what would happen if the ball was not thrown horizontally (using the negative values of □ that satisfy the above equation).

PROGRESS REPORT ON ESTABROOK PROJECT - APRIL, 1964

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PROGRESS REPORT ON ESTABROOK PROJECT - MAY, 1964

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2944-66

PROGRESS REPORT ON ESTABROOK PROJECT - JUNE, 1964

Classes at Estabrook continued until June 18, and the last classes in our units were held June 17. In grades 3 and 5, having come to a natural stopping point in the main units before the end of the term, new material was introduced in the last two weeks.

In grade 1 we played the "paper-stone-scissors" game and examined it as an example in which our 3-stick rule did not work. There were frequent requests to repeat the game. We developed the ½ points on the stick line, first by sandwiching a point (it is < 3 and > 2) then by defining $\frac{1}{2} + \frac{1}{2} = 1$ and cutting a paper stick until it satisfied that criterion. All the ½ points were then constructed by addition and subtraction. The 1/3 points were defined analogously.

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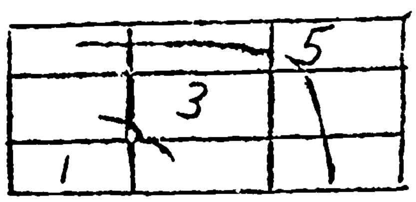

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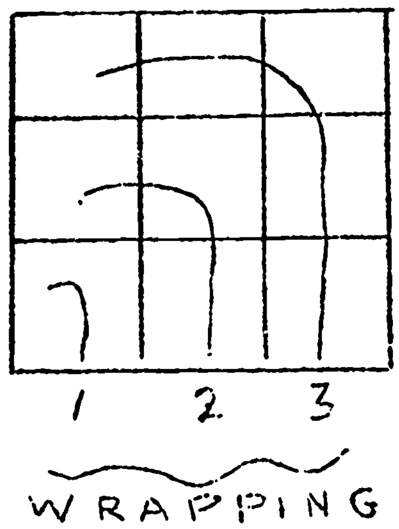
REPORT ON ESTABROOK PROJECT FOR OCTOBER, 1964

The Kappa team class discussed sequences such as 2n, 2n - 1, 5n, 2^n, $\frac{1}{2^n}$, $1 - \frac{1}{2n}$, and $(-1)^n$ with regard to their pattern, the "mathematical sentence" for this pattern, and convergence properties. The question of comparing factions arose in this context. They had a chance to invent sequences as well as discover patterns in given sequences. A square was built up of successive "wrappings" of small squares as shown



and the sequence Un = 2n - 1 of little squares in each wrapping n, and the sequence $w_n = n \times n$ of little squares in the big square was observed. By trial and by geometrical discussion they obtained the above relations between n and Un and n and w_n. Fianlly, by comparing the two ways of constructing the big square (n wrappings or n rows of n squares), they found that $1 + 3 + 5 + \ldots + (2n-1) = n \times n$. Concurrently the class teacher, Mrs. Moynahan, developed linear "open sentences" and graphing.

In the Delta team class the teacher, Mr. Barnes, developed the algebra of rotations and flip of equivalent triangles along the lines of last spring's work.

There were less classes than usual due to the Thanksgiving holiday.

The students were given work sheets on which they had to find n (the number of wrappings), U_n (the number of boxes in a wrapping) and W_n (the number of boxes in the square array) given any one of them. They showed facility in obtaining the answer for large n (e.g. 79) as they remembered the pattern. They were able to explain the geometrical reasons for the numerical pattern, and some were able to construct the mathematical sentences that represented this pattern. They started with the relation $U_1 + U_2 + \ldots U_n = W_n$, and by substitution from $U_n = 2 \times m-1$ and $W_n = n \times n$ obtained the relation $n \times n = 1 + 3 + \ldots + (2 \times n - 1)$. They tested this for many number and agreed that as it was a relation between numbers it did not depend on the geometry from which we derived it. We set out to prove it from the rules of arithmetic and logic. We discussed mathematical induction in terms of the analog of a row of bricks standing on end such that when one falls it pushes over the next. They seemed to appreciate the process well, including the need to prove the statement true for n=1. However in applying induction to our series relation their understanding of arithmetic laws was so poor that we did not succeed. We have started a discussion of the C.A.D. laws in terms of arrays to remedy this weakness.

At the end of the month a first class was held with Mrs. Moynahan's fifth grade class of above average students. We approached the commutative and associative laws of addition through the "sweeping of piles of trash" dialogue used by Max Beberman.

The class was given its own example to work out with instructions to use whatever shortcuts they could think of. Those in the class who used the symmetries (3/4 of the class) all finished before those who did not observe and use the symmetries. This a bright and interested class.

21 21	10	11	21 21
5	3	2	5
16	7	9	16
21 21	10	11	21 21

REPORT ON ESTABROOK PROJECT FOR DECEMBER, 1964

Instructional work at the second-grade level was started this month. Several meetings were held with Miss Marie Mortimer (who taught last year's first-grade project) and Miss Kay Dillmore, who supervises mathematics in the ϕ team (first and second grade).

It was decided that we would go through the material tried in first grade (only four children in the new class were in the first-year group), expecting that this bright group of second graders would do the material faster, and could be pushed to a use of strategy and deduction with inequalities, with which the first graders had trouble. These children have a background of set theory from the "Greater Cleveland" sequence, so that inequalities of sets and lengths may be compared and the cardinals can be represented as a subset of the number line. We intend as intensive development of fractions and negatives, to discuss crossed-number lines and multiplication, and to develop the C.A.D. laws using two and three dimensional arrays. We also will try some units on geometry and symmetry, the latter using Miss Marion Walters' work.

Instruction will generally be by Miss Mortimer. I will often observe and occasionally teach. On December 18, the first class was held with me instructing. We discussed the comparison of numbers, elephants with giraffes, etc. Many bases of comparison were discussed and shown to lead to different results. We then asked for an operational definition of longer and shorter sticks and immediately obtained the "put one end of each stick together and see which stick has a piece left over at the other end" definition. As they had "sets" I asked them to compare two sets and find out which one had more elements <u>without counting</u>. After discussion and trials, they came to the one-to-one check-off, the set having elements left over being the bigger. Throughout, we several times reversed the question from "which is bigger" to "which is smaller" to develop the $a > b \supset b < a$ rule.

The sixth grade class moved quickly with "Madison Project" material on true, false and open sentences. We discussed identities and anti-identities (always false for legal substitution) and found we could always make the latter by "spoiling" the former. Collected the $\Delta = \Delta$ and $\Delta + \square = \square + \Delta$ identities among a few others. In the next session, I came in while they were having difficulty with a word problem. We discussed it from the point of view of translating English sentences into math sentences. This key problem would seem to be soluble by more explicit treatment in class. The fifth grade class was given some of the same work by Mrs. Moynahan.

In the fourth grade Mr. Barnes has completed the rotational symmetry work and has taken translational symmetry to the point where he can begin work on vector addition in two dimensions.

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PROGRESS REPORT ON ESTABROOK PROJECT - JANUARY, 1965

Each of the fifth grade and sixth grade groups had four parallel lessons each. The last two sixth grade lessons were taught by Mrs. Moynahan. It is planned to continue in this way with the two classes in steps. This will permit the comparison of similar work by a slightly younger but somewhat more able group with that of the sixth grade students. However, when we later deal with limits, it is to be remembered that the fifth grade students did not have the work with sequences of October and November.

During the month both groups identified the C.A.D. laws, the properties of 0 and 1 and several other identitites. They also examined the proving of some identities by the use of others. They agreed (by examination of cases) that these were identities for positive and negative rationals. We then examined the connection between the C.A.D. identities over the positive integers with the symmetry properties of one, two, and three dimensional arrays of dots. During the last session of the month they were asked to extend the comparison to rationals (positive). For the commutative law of addition they came close to the placing of line segments end to end. Their first reponse (for fractions totalling less than one) was to shade in slices of a pie. They indicated the commutativity by considering rotations or inversions of the pie. That was an unexpected, welcome response.

The fourth grade, under Mr. Barnes, developed their work on discrete translational symmetries of infinite arrays of dots into the vector concept in a plane. Vector addition and vector components were treated. The three dimensional problem was also discussed.

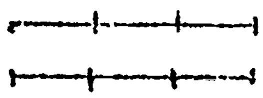
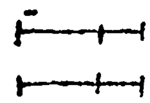
Miss Mortimer taught three of the lessons of the second grade (above average) class, and I taught one. The students proceeded at a rapid pace. The method of comparing sticks was likened to the comparison of sets by one-to-one correspondence. The "two stick rule" ($A > B \Rightarrow B < A$) was discussed in this context. The colors on the sticks were switched to show that the rule was independent of labelling. The "three stick rule" ($a > b, b > c \rightarrow a > c$) was worked out through the "warrior game". The Green Knight and the Red Knight were belligerents who kept the size of their weapons (green and red sticks) secret from each other. However, a neutral, the White Knight, was allowed by either belligerent to compare his weapon with theirs. The class then decided if, with this information, they knew which of the belligerents had the bigger stick. They quickly decided that the neutral's stick had to be bigger than one of the red or green, and smaller than the other, to be of help This evolved to the requirement that when the inequalities were written so that the comparison stick was "sandwiched" the pairs had to be in the same order to result in a "three-stick rule". $a > b > c \Rightarrow a > c$. "If (a,b) and (b, c) are in the same order, so is (a, c)". They were given work sheets to check their adeptness at recognizing and applying the two and three-stick rules. They were then asked to construct a four-stick relation (after making some comparisons among four sticks). After setting up the relation $a > b$, $b > c$, $c > d \Rightarrow a > d$ they then found the "three-stick rule" within that statement. They were able to show, by applying the "three-stick rule" twice, that $a > d$ followed from the above. The generalization to five (and more) stick rules followed easily. They enjoyed the example of the "paper, scissors, stone" game as a case in which the order relations (the three-stick rule) did not apply. They seemed to recognize the contrast clearly. They have now started to apply the order relations to a game of strategy. In this game (used at the Morse School sessions last summer) an unseen stick has its length located among five given sticks by asking the fewest $<$, $>$ questions.

The demonstration by Marion Walter of the mirror cards and the subsequent plans of all the teams for their use have been described separately.

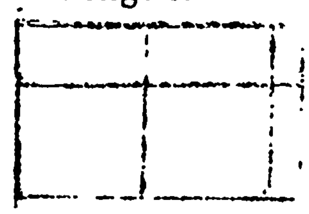
2947-66

PROGRESS REPORT ON ESTABROOK PROJECT - FEBRUARY 1965

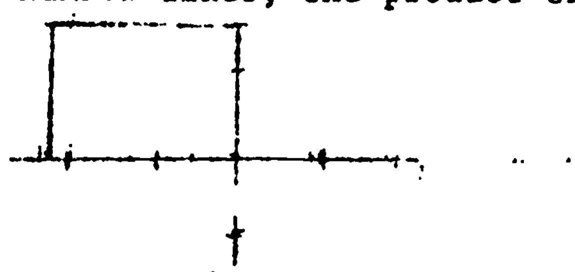
The fifth grade class, having found that dots were only convenient as a representation of the positive integers, developed line segments as a representation of any real number (they worked with rationals). The sign was represented by the direction of the segment from the starting point on a fixed line. They were then able to represent the associative and commutative laws of addition for arbitrary real numbers. To represent multiplication they first tried several rows of line segments:

2 X 3  2 X 1½ 

But this would not work for 1½ X 2 or 1½ x 2/3, etc. They then made a rectangle with sides of the proper length:

1½ X 2¼ 

This representation was useful, not only as a symmetry argument for the commutative law of multiplication, but also in improving the classe's understanding and performance in the multiplication of fractions. By drawing the rectangle on crossed number lines, the product of negative numbers was included:

1½ X (-2¼) 

They observed that negative products turned up in the second or fourth quadrants. Having agreed that a negative times a negative should be positive, on the basis of a rate X time = distance discussion, they found that positive products turned up in the first or third quadrants. The sixth grade goup developed parallel material.

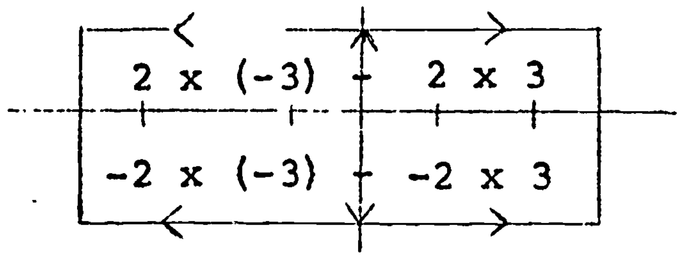
In the fourth grade the effect on a vector of rotating the co-ordinate system was explored. They found that the components changed; but that of course the length of the vector or its angle with respect to any other fixed direction did not change. They were just beginning to explore the problem of expressing the invariant length in terms of the variable components.

The second grade children played a version of the strategy game (find the position among five sticks with the fewest questions) in which itwas not possible, with luck, to locate in less than three questions. Criminals (of all six possible heights to fit among five sticks) were able to secretly pass a diamond between them. The detective could ask three $\langle\ ,\ \rangle$ questions. Before answering the mastermind for the criminals could give the diamond to any of them, but then a truthful answer to the detective's question was required. After each question the eliminated set of criminals dropped from the game. Before answering the next question the mastermind could place the diamond with any one of his remaining criminals. Several students quickly learned the secret of the three questions. We then allowed masterminds from the class to challenge detectives until most of the class discovered the secret for themselves. Several were able to explain the strategy well. The game was extended to seven and eight sticks. In the latter case they quickly found out that they needed one more question. Although the strategy of splitting in two was well understood by the class, it was not clear to what extent they realized the connection to the "three stick rule". In the last session they developed methods of adding sticks, and of marking off lengths on paper to facilitate comparison. They predicted and then checked that with $A < B$ and $B < C$, then $A + B < A + C$. I was the instructor for the first session and Marie Mortimer for the other two sessions of the month (there was a holiday week).

Two Kappa team classes have been using the mirror cards extensively, with satisfaction.

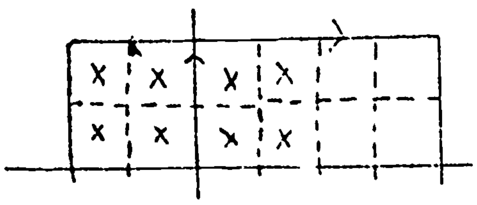
March, 1965

The fifth grade students were asked to notice the direction of motion when constructing the rectangles denoting products. The convention was to mark off the first number on the vertical line starting from the origin, and then to proceed in the horizontal direction for the second number.

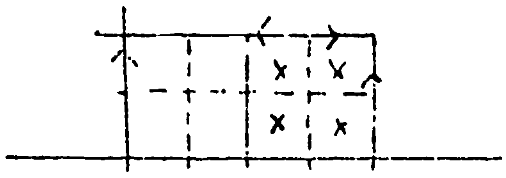


They easily observed that products which were positive according to our previous discussion went clockwise and vice-versa. They seemed impressed by this and willing to include these "signed areas" as a <u>reason</u> for (-2) x (-3) = + 6. They agreed that we now had several situations in which (-) x (-) = + was the convenient convention.

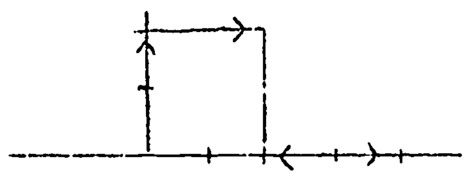
Working at their desks they invented various rectangular representations of quantities like 2 x 4 + 2 x (-2), and counted up the answers by cancelling out equal areas of opposite circulation.



or



In both of the above solutions, the x's mark boxes cancelled out, leaving the answer of 4 units. In the second case, a technique was developed by a student of overlaying contributions of opposite circulation to accomplish the cancellation directly. The class also applied these techniques to products of fractions. For comparison they constructed 2 x (4 -2)



They also graphed (2½) x (1½) + (2½) x 3/4 and (2½) x (1½ + 3/4). From these the distributive law was abstracted.

They developed skill in determining the fractional size of rectangles by constructing equal rectangles to fill the unit square. They then abstracted the definitions 4 x ¼ = 1 2 x ½ = 1 3 x 1/3 = 1 etc., and explored the possibility of finding products of fractions starting from the above definitions and the C.A.D. laws. For example: 1/2 x 1/3 = ? Start with 2 x 1/2 = 1 and 3 x 1/3 = 1. Then 1 x 1 = (2 x 1/2) x (3 x 1/3) "as we can replace 1 by something else equal to it." But 1 x 1 =1 and (2 x 1/2) x (3 x 1/3) = (2 x 3) x (1/2 x 1/3) by C and A. Therefore, 6 x (1/2 x 1/3) = 1 and it follows from definition that 1/2 x 1/3 = 1/6. Similarly they found that 2/6 = 1/3 using 2/6 = 2 x 1/6. They discussed 2/3 x 3/4 and 3/4 x 5/8 but were not prepared to co-ordinate enough definitions and substitutions.

The students now had enough skill with negative and fractional numbers to proceed with the graphing of curves. A discussion of the motion of objects thrown along the floor and through the air was introduced as motivation for describing curves by mathematical sentences. After much throwing of blackboard brushes and chalk several conclusions were reached: On a smooth horizontal surface objects tend to move in straight lines. They go further if they are smoother or roll well. Objects thrown through the air curve downward "because of gravity." They always land in about the same time if thrown horizontally at different speeds, though they travel different distances. Straight line motion obtained by throwing straight up was different from motion on the floor in that the object slowed, reversed, and speeded up.

The sixth grade also worked on multiplication of negatives and fractions.

I taught a session in the fourth grade for the first time this year. The vectors (3,4) and (3, -4) were noted by symmetry to have the same length. By measurement the length was found to be 5 units and some volunteered the relation $3^2 + 4^2 = 5^2$. Could we obtain the length from the co-ordinates in the second case? $3^2 + (-4)^2 = 5^2$. The class was divided on $-4 \times (-4) = \pm 16$. As in the fifth grade we graphed $(\pm 2) \times (\pm 3)$ and discussed walking at different rates to justify the multiplication conventions.

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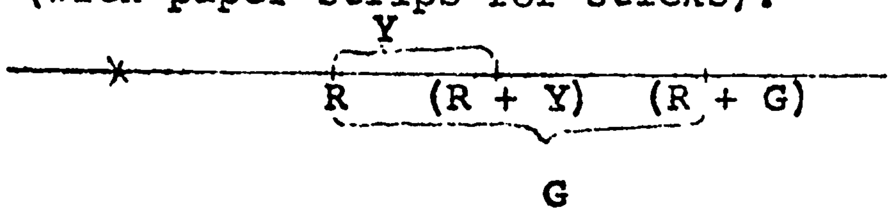
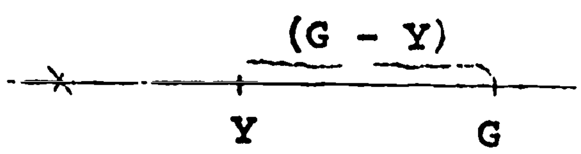
The other sessions, given by Dick Barnes, developed the class understanding of multiplication of fractions using rectangular plotting. That skill is required in order to have some flexibility in rotating co-ordinates while keeping vectors (and vector pairs) fixed. The class showed, unexpectedly, an ability to scale, ordinate and abscissa separately, comparing effectively to unit rectangles.

The second grades compared two sticks and then were asked to consider the addition of an unseen stick to each. They showed a quick response and understanding of variations on $G > Y \Rightarrow (R + G) > (R + Y)$ with different size R. While checking their predictions they developed facility with addition on a "Stick Line" and its notation, both at the blackboard and on mimeographed sheets (with paper strips for sticks).



They also compared (G + R) with (Y + R) and (G + R) with (R + G) mentally, then checked.

They were asked how they would find out how long a stick was left over if they took away Y from G. They did the obvious with sticks, and were then asked to use the stick line. The response was both



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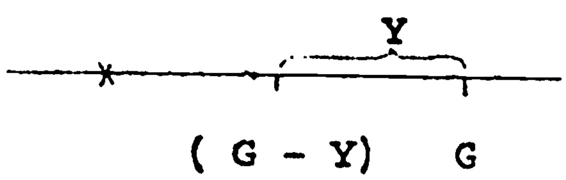
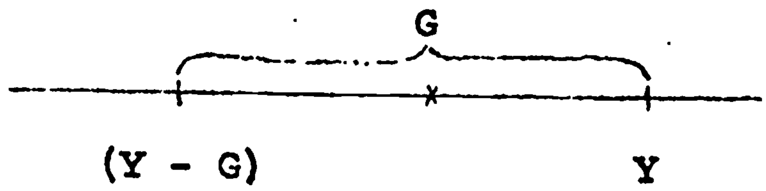


They observed that the second way was more in accord with their method of addition, i.e. the second stick from the end of the first stick. After several such additions and subtractions, the class was asked to subtract a larger stick from a smaller. Many voices were raised to say "you can't do it" but the majority of the class was not impeded by that consideration. They were only told to try proceeding as before. Then, from different parts of the room, "I've got it". The whole class eventually discovered,



What is different? "It falls on the other side of x". Using your fingers compare the distance x to (G-Y) with x to (Y-G). "They are the same length". They did other examples including $[(G - R) + Y]$ and $[(R + Y) - G]$. They compared (Y - G) with opp. (G -Y) and with (-G + Y). (By opp. they meant flip over about x.) A reminder of the transitive pattern in $a > b,\ b > c \Rightarrow a > c$ was required before they could make the parallel.

$$\text{opp } (G - Y) = (-G + Y),\ (-G + Y) = (Y - G) \Rightarrow \text{opp } (G - Y) = (Y - G)$$

We then considered lengths of chalk; we receive this length for the week and use varying amounts each day. How much do we have left over at the end of one week? How much do we have to borrow at the end of another week? "We borrow a piece big enough to bring us back to the starting point from where we ended up on the negative side". The class was then given the length received and the length used each day. They were to report when they reached a day on which it was necessary to borrow chalk. About half the class was correct the first time around. Marie Mortimer taught three of the sessions. I taught one and observed another.

One afternoon about 18 volunteers from the second, third, fourth and sixth grade classes stayed after school to be introduced to mirror cards by Marion Walter. This session was requested by the teachers who had been using, or intending to use, the cards. They wished to observe Miss Walter's approach. She worked with about six children at a time (according to age group). There was a large amount of enjoyment and successful manipulation. A few took some time to see the use of blocking with the mirror as well as doubling what was left. One girl did a rapid and accurate job on the most advanced sets.

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April, 1965

PROGRESS REPORT ON ESTABROOK PROJECT

In the second grade Miss Mortimer continued the discussion of chalk lengths obtained, used and borrowed. This time the children started off the week in debt (on the negative side of the starting point). They added and subtracted the different lengths (G, Y, and R sticks of odd sizes) corresponding to chalk obtained from the office and chalk used. The children made it known when they were first "out of debt." The concept of using a fixed amount of chalk each day was introduced and the children asked to mark off the total they would have used by Monday, Tuesday, Wednesday, etc. When asked how they would mark these positions they responded with M, T, W, T, F; 1Y, 2Y, 3Y, 4Y, 5Y, (the yellow stick was used) and also 1, 2, 3, 4, 5. The starting point became 0. They marked off the various amounts they would have to borrow and suggested -1, -2, -3, -4, -5. They had thus developed a "number line" in part. They were asked to use this "number line" in solving such problems as, "Two pieces of chalk were delivered Monday. If one piece was used each day, how much had been borrowed by Friday?"

They were asked how many halves were in a whole. They were then asked how they would find half of a standard or unit piece of chalk. One child suggested, "folding in half," but another said this would not be done with chalk and suggested breaking and finding, by trial and error, a piece which would add to itself to reach 1 from 0 on the number line. This was taken as the standard. The students cut strips of paper to fit this criterion. They were of course allowed to use symmetric folding to decide where to cut, if they wanted to. With this 1/2 length they found the positions of 1/2, 1-1/2, -1/2 etc., on their number line. They were familiar with the notation 1/2, but were still developing the nomenclature and symbolism for the others at the end of this month.

Discussion with Miss Kay Dillmore has developed plans for more intensive work with these students in May and June, in which they would work throughout the week on materials connected with the "Goals" report.

In the fourth grade (perhaps it is a good time to be reminded that half of these students are actually third graders within the team) Dick Barnes

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and I each taught several classes. The class found the result of 1/2 x 1/3, of 1-1/2 x 3/4, etc by rectangular plotting. To my surprise, they used independent scales on the horizontal and vertical axes and had no trouble in relating to the unit rectangle. They were then asked how many facts about the multiplication of fractions were there to be found from such contructions. They said there would be no end, so they were posed the problem of finding a short list of facts from which the rest could be found.

They were asked how many number facts they needed to multiply any whole number in base ten by any other. The first answers were "9 x 9" and " 55, because some of the 9 x 9 are the same by the commutative identity". They then decided that they used those facts plus the commutative associative and distributive identities in long multiplication of whole numbers. They also considered the much smaller list needed in binary notation. To obtain the power with fractions that they had with integers they first discussed the definition of fractions. The verbal definition "four pieces of quarter size make a whole piece" became 4 x 1/4 = 1, etc. With that definition, the whole number results, the C.A.D. identities and the properties of 0 and 1 they then were asked to find the result of 8 x 1/4. They proposed both

8 x 1/4 = (2 x 4) x 1/4 = 2 x (4 x 1/4) = 2 x 1 = 2 and

8 x 1/4 = (4 + 4) x 1/4 = 4 x 1/4 + 4 x 1/4 = 1 + 1 = 2

and were able to identify the parts of the short list that were used at each step. In a similar way, they discussed 2 x 1/6 = 1/3, 1/2 x 1/3 = 1/6 and 2 x 2/3 = 1/3, etc. (in the latter using 2/3 = 2 x 1/3). They had increasing difficulty. But one student in the class proposed methods such as

(1/2 x 1/3) = ☐ 2 x (1/2 x 1/3) = ☐ x 2 1/3 = ☐ x 2

3 x 1/3 = ☐ x (2 x 3) 1 = ☐ x 6 Therefore ☐ = 1/6

This led to a discussion of keeping the "balance" in an equation and the laws of equality or "balance."

In the fifth grade after a further discussion of the physical paths of thrown objects, it was suggested to the class that having mathematical sentences that described these paths would help us in the understanding of these motions. The students were asked to suggest mathematical sentences that could be examined. They presented $\triangle \times \diamond = \diamond$ $\triangle \times \square = \square$ and $\square \times \triangle = \square$ and realized that these would have the same true pairs as only the frames used were different. They suggested $1 \times \triangle = \triangle$ and $0 \times \triangle = 0$ and noted that these were identities. They also suggested $\triangle = \square = \square$ and $-1 \times \square + \triangle = \square$ Almost everyone in the class was quick to present true pairs for the first of these and stated " $\triangle$ must be zero" and "any number at all for $\square$." They had more trouble with the second. Although when checking $-1 \times \square$ they obtained the correct result, this part of the sentence seemed to inhibit their guessing of true pairs. They were given graph paper and asked how they would record the true pairs. "Put down crossed lines and number them like number lines. For (0,1) put a mark at 0 up and 1 across." They quickly plotted their true pairs for $\triangle \div \square = \square$. "It is a straight line going across" was easily noted with interest. They also found that any other point on this line was a true pair for the same sentence. Turning their attention back to $-1 \times \square \div \triangle = \square$, could they make the guessing of true pairs easier? Would adding equal amounts to both sides of the sentence make true pairs false or false pairs true? They said that the balance would be maintained for the same true pairs. What could be added to the left hand side so that only $\triangle$ remained? What did they then have to add to the right hand side so the true pairs would be the same? They obtained $\triangle = 2 \times \square$ for which all members of the class were easily able to find examples of true pairs, including negative and fractional numbers. They then checked that each of these true pairs satisfied $-1 \times \square \div \triangle = \square$ They plotted these true pairs as points, again observing the straight line result.

2905-65

EDUCATIONAL SERVICES INCORPORATED

30 June 1965

FROM: CAMBRIDGE CONFERENCE ON SCHOOL MATHEMATICS

Enclosed is Earle Lomon's last report for the scholastic year 1964-65 on the Estabrook School Project which he conducted in Lexington. The report covers May and June, 1965.

M. R. Gatto

June 19, 1965

PROJECT REPORT ON ESTABROOK PROJECT FOR MAY AND JUNE 1965

This is the final progress report for 1964-65, although there will be one more lesson at each grade level. It is expected that a full and final report will be available by August 15, 1965.

Since the beginning of May the second grade class has been instructed by Miss K. Dillmore two or three times a week using C.C.S.M. material, in addition to my weekly contact. She concentrated on multiplication by the use of arrays and of rectangular areas. Arrays of dots on blank paper were used first. As the numbers became larger the students were given 1/2 inch graph paper and were asked, for ease of counting, to mark off -1, 0, 1, 2, etc. up, and also across. This elicited from several students "This is just the same as using two number lines crossing each other." They marked off rectangles, according to the numbers in a product, and counted squares quickly, by twos, fives, etc. They were easily able to do multiplications such as 12 x 17 in this way. They noted that altering the order of the numbers only rotated the rectangles, so that the number of squares had to be the same. They also looked for missing factors - how many rows of 3 squares were needed to make 12. Asked for the products of 10 x 2, 10 x 4, 10 x 10, etc., they realized that these represented a class they could answer readily. When they were then asked to graph 23 x 12 marking off blocks of easily determined numbers of squares, many made maximal use of products of ten. They marked off the two 10 x 10 blocks, the 10 x 3, the 2 x 10 and the 2 x 3 blocks. They readily multiplied integers less than 100 in this way. They used the same rectangle construction for multiplying fractions, symmetrically subdividing the unit square into a sufficient number of rectangles to determine the fractional area. After the above exposure, the Greater Cleveland exercises on multiplication were found to be too easy for the students.

The development of fractions was continued on the number line, during the session I taught. A variety of games of strategy between paired opponents were played with the line marked in half units. A typical game was starting at -1, each player in turn could choose to add 1 or 1-1/2 or subtract 1/2. The last choice was not permitted twice in a row. The first player to reach +2 won. This type of game, which they liked to play at length, quickly developed their facility for adding and subtracting fractional numbers. They defined 1/3 (three of these made up the unit interval), marked off the line and played a game as above. They then easily extended to defining and working with 1/4, 1/5, etc. In a game permitting jumps of 2/3, 1 and -1/2 they found that they had to divide the unit into sixths to handle all results, and found relations such as 2/6 = 1/3. They answered many questions on inequalities of fractions such as which is bigger 3/5 or 4/5, 1/10 or 1/11, 8/9 or 9/10? Only a few students had some trouble with the relative size of 1/10 and 1/11, for instance. Other students explained that the pieces had to be smaller to get more of them into the unit interval.

For the last two weeks new material was introduced. The bulk of the class worked with Marion Walter's unit on the putting together of triangles, edge to edge. They observed the uniqueness of shapes made in this way using two or three triangles. They discovered that you could get new shapes if the laying of one triangle on top of the other was permitted, while maintaining the edge to edge rule. They then all had time to find the three shapes that could be made out of four triangles, and found they could fold these into tetrahedra or square pyramids. With the rest of the class Miss Dillmore worked through the beginners into the intermediate sets of Mirror Cards. She has had and will have further Mirror Card sessions with different members of the class. My last session of the class will be occupied with Marion Walter's unit of arranging up to five squares.

The mixed third and fourth grade class completed their discussion of fractions by considering their addition. The distributive law together with their previous results for the multiplication of fractions was used to deduce the addition results. It was necessary to return to the graphical representation of the distributive law as the class had largely forgotten its content and use.

We then returned to the "push" or vector $(\square, \triangle) = (3, 4)$. They found by measurement, that its length was 5. When asked if there was a relationship between its length and its components they suggested and checked $3^2 + 4^2 = 5^2$. After an abortive attempt at drawing co-ordinate lines on top of the mimeographed system, transparencies were made for each student. Rotating the transparencies over the mimeographed sheet by 45°, they measured $(\square, \triangle) \approx (4\text{-}7/8, 7/8)$ and checked that $\triangle^2 + \square^2 \approx 5^2$ even though $\triangle$ and $\square$ had changed greatly. They then looked at 90° $(4, -3)$, 180° $(-3, -4)$, and 270° $(-4, 3)$ rotations and checked $(4)^2 + (-3)^2 = 5^2$. We reverted briefly to oriented areas to affirm $(-3)^2 = +9$. They then successively oriented their transparency axes so that the $\pm \square$ and $\pm \triangle$ axes fell along the direction of the "push", obtaining $(5,0)$, $(-5,0)$, $(0,5)$, and $(0, -5)$, with the trivial result $(5)^2 + (0)^2 = (5)^2$. They are now choosing their own rotation arbitrarily, will measure $\square$ and $\triangle$ with a ruler, and check the relation $\square^2 + \triangle^2 = 5^2$.

The fifth grade class considered the relations of the true pairs for $-\square + \triangle = \square$ and $\triangle = 2 \times \square$. Adding $\square$ to both sides would keep the "balance" they agreed. They found many true pairs for the second form and checked that they were still true pairs for the first. They happily observed that when these true pairs were plotted as points they fell on a straight line. They checked that other points they chose on the line were also true pairs. Their graph showed that when they went across by 1 they had to rise by 2 to keep on the line. In fact they could go across by different amounts from different places, but the ratio of up to across was always 2. From the nomenclature familiar to them for hills this was called a slope of 2 in 1, or simply a slope of 2. They were asked for sentences whose true pairs would plot as steeper hills, they suggested $\triangle = 3 \times \square$, $\triangle = 100 \times \square$, etc. For gentler slopes they suggested $\triangle = \square$, $\triangle = 1/100 \times \square$, etc., some were plotted. They were asked how to get a hill that went down when moving across to the right. They suggested $\triangle = -\square$, which was checked, and the gentler and steeper "down" hills were proposed. They went through a sequence of smaller and smaller slopes to arrive at $\triangle = 0 \times \square$ or $\triangle = 0$ for a horizontal line.

For a straight up hill they arrived, after a sequence, at $\triangle = \infty \times \square$. They found that this only gave $\triangle = \infty$ (for $\square$ any non-zero number) or $\triangle$ ambiguous (for $\square = 0$). To find a better alternative we compared $\triangle = 100 \times \square$ with $1/4 \times \triangle = 25 \times \square$, with $1/100 \times \triangle = \square$, etc. Keeping the "balance" by multiplying both sides by the same amount was used as a procedure of getting from one to the other, but also true pairs were checked for their transferability. They then noticed the sequence $\triangle = \square$, $1/10 \times \triangle = \square$, $1/1000 \times \triangle = \square$ etc., represented lines that were more and more vertical. This led them to $\square = 0 \times \triangle$ or $\square = 0$ as a matter of course.

The students were asked for a line that would lie above $\triangle = 0$ and would still be horizontal. After one student suggested $\triangle = 2$ other suggestions such as $\triangle = 1{,}000{,}000$ came quickly. It was then easy to elicit $\triangle = -4$, etc for lines below $\triangle = 0$. In a similar way we discussed $\square = 3$, etc. for lines to the right and left of $\square = 0$.

Mrs. Moynahan had an extra session with a volunteer group of nine students from this fifth grade class, discussing the turning and shifting of lines. In a lesson with this smaller group, I asked them to plot the points (-2, 0) and (2,2) and to draw the line through them. By drawing various right triangles with segments of the line as hypotenuse, they found the slope. For a sentence with that slope they wrote down $\triangle = 2 \times \square$. They were then asked what they would have to add to that sentence to make the points on their line true pairs. Several found $\triangle = 2 \times \square - 2$ independently. When then asked to write a sentence for a line going through (-3,0) with a slope of 2 they quickly obtained $\triangle = 2 \times \square - 3$. Given more pairs of points they readily found the slopes, including cases of negative slopes. In looking for the slope it was suggested to them that they use the triangle that spans the two given points. They found that the length of the sides of this triangle were given by the difference of the point co-ordinates, so that they could find the slope without plotting the line or the points.

The students were then challenged to replace the number representing the slope by something that would change as the number in $\square$ was altered --

so that the slope would change with position. The first suggestion was $\triangle = \diamond \times \square$. While telling them that this answered the question in a good way, I pointed out that our planer plot only permitted two frames. These frames could appear more than once however. The next student suggestion was $\triangle = \triangle \times \square$. They found that this had two different sets of true pairs; the line $\triangle = 0$ and the line $\square = 1$. There was certainly a big change of slope in going from one to the other. What else? They then suggested $\triangle = \square \times \square$. They found true pairs like (0,0), (1, 1), (2, 4), (1, -1), (1/4, -1/2), etc. They plotted these and decided it looked like a cup, etc. I asked them what it looked like upside down. They suggested a bridge and other things, and agreed it looked like the arc of a piece of chalk I threw through the air. The suggested $-\triangle = \square \times \square$ to turn the curve over to look like the chalk arc. One student said that it was too pointy for most arcs obtained. They were then asked how to make $\triangle$ change more slowly as $\square$ increased. Trying $-\triangle = 1/16 \times \square \times \square$ they obtained a satisfactory result. The question of finding a slope at a point of this parabola will be pursued in the remaining session with this group.

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